A Novel Line Fractal Pied de Poule (Houndstooth)

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Abstract

In earlier work we proposed a fractal pied de poule inspired by Cantor's dust, building on a mathematical analyis of the classical pied de poule pattern. Now we propose another fractal pied de poule implemented as a single line. Instead of blocks which get fragmented into smaller and smaller blocks, we begin with a single continuous line which is expanded by adding more and more nested zigzags. Although the former approach takes a two-dimensional starting point and the latter a one-dimensional starting point, the resulting fractal dimensions are comparable (also depending on the type of the original pied de poule being fractalised). We calculate the fractal dimension and develop a fashion item based on the new pattern, to be shown at Bridges.

Introduction

After our earlier mathematical analyis of the classical pied de poule pattern [1] and the invention of a fractal version of it [2] we felt that there was not much more to be explored in houndstooth. Then in 2014 we presented an innovative line fractal [3] (not pied de poule) for which we found turtle graphics [4] to be a useful tool. In contemporary fashion, pied de poule is still very much alive. It is an important cultural and mathematical phenomenon [5, 6]. So at some point we asked ourselves whether it would be possible to construct a line fractal for pied de poule.

Figure 1: *Christian Dior's 2012 design(left), detail of the same, chain of basic pied de poule figures of type two and type one (with zigzags) and with recursive zigzags (rightmost figure).*

We got inspiration from one of Dior's designs of 2012, Fig. 1 (a) which we also mentioned in [1] (mainly noting that the basic figures are isolated and appear to fly out). But zooming in to the same design, as in the second picture of Fig. 1, we now noted something else: that each basic figure was in fact a kind of zigzag line. We already knew that inside a classic pied de poule pattern, the black basic figures are connected and thus form chains, as shown in Fig. 1 (third picture). Zooming out, such a chain could be considered a kind of line. Perhaps the zigzags could be chained and at the same time the line drawing could be done zigzag-wise in a recursive manner (last two pictures).

From [1] we recall that there is not just one pied de poule, but an indexed family, one pied de poule pattern for each integer $N > 0$. This is shown in Fig. 2. We deployed three types of mathematical tools for our analysis: (1) compact formulas to describe the twill binding and the color patterns of the warp and the weft, (2) Heesch and Kienzle's theory to describe how the basic figures are tessellated, and (3) turtle graphics to describe the contours. In Fig. 2 we show the pied de poule patterns for $N = 1$, 2, 3, and 4. Therefore we decided that it would be better not to look for just a single fractal but rather for a family, one for each N.

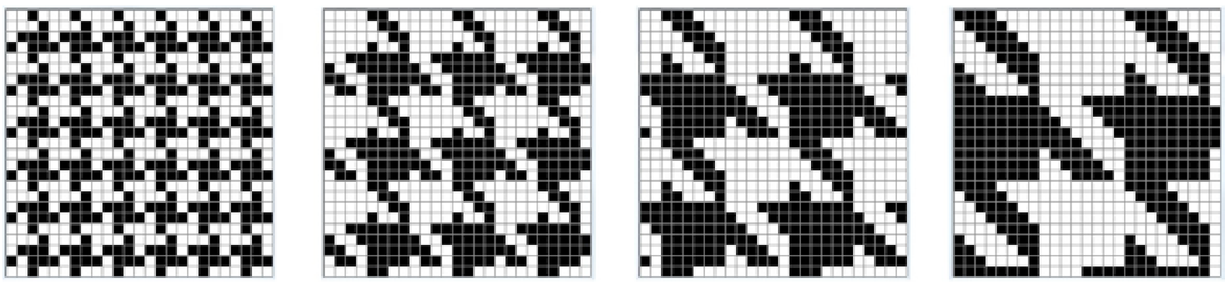


Figure 2: *Successive pied de poule patterns for N=1,2,3,4 from [1].*

Since we aim at a line fractal design, we also have to mention some of the known examples of line fractals, see [7], such as the Koch curve, the dragon curve, the Sierpiński arrowhead curve, the Hilbert curve, Kaplan's smooth self-similar curves [8] and our own warp knitting fractal [3].

Requirements

So we set out to search for a fractal line which would embody some of the key principles from [1] and would be recognisable as a pied de poule pattern. The initial explorations were promising, as demonstrated in Fig. 1. But except for the case $N = 1$ the explorations appeared difficult and messy as we found several fractal lines which were piede de poule-like, but had a deficiency of one or the other kind. We had to structure the process first, defining more precisely what would be a good fractal line pied de poule. Since there is not a single classical piede poule, we aimed at a family of fractal lines, not just a single solution. In summary, our requirements are that it has to be pied de poule-like, recursively tessellated, parameterized, generic and continuous. We explain each of the requirements in detail now:

- Pied de poule-like: for each N the line should resemble the classic basic pied de poule as formalised in [1], which in turn originates in the weave and in the block patterns of the warp and the weft ("N-over, N-under" weave and patterns of $2N$ white threads followed by $2N$ black threads).
- Recursively tessellated: the number n is the recursion level such that the figure for level n should look like a tessellation of figures for type $(n-1)$. It should be tessellated in the characteristic pied de poule style, which implies that the empty space between the figures has the same form as the figures themselves (although it would be okay for this same-ness only to appear as a limit case).
- Parameterized: we demand a family of lines, one for each N (which can be approximated for each n).
- Generic: the figures should be described or generated by a generic recipe with a minimum of ad-hoc tricks and which works the same for each N and each n.
- Continuous: each line is a continuous line without jumps.

Initial explorations

We found a line fractal based on a zigzag inscribed in the basic figure of the classic pied de poule for $N = 1$: the long diagonal lines are implemented as a sequence of recursive zigzags, whereas the short lines, which serve for shifting to the next diagonal, will be done just straight, as indicated in Fig. 1 (right). The recursive zigzags of adjacent lines can be aligned such that they form a pied de poule-style tessellation. We implemented this using turtle graphics (which also has been an important tool in our earlier works such as [1, 2, 3, 4]). The turtle graphics command sequence for the downgoing diagonals has to be reversed with respect to the command sequence for the upgoing diagonals. In Fig. 1 (right) the turtle begins at the lower end of the entire chain.

The fractal line pied de poule of Fig. 1 (right) was quite satisfying and we set out enthusiastically to generalize this to larger N values. This turned out difficult. To illustrate the nature of the difficulties we refer to Fig. 3. From the classic figure taken from [1] we see that the number of diagonals is always even. But it seems we have to begin with an upgoing diagonal and we have to end with an upgoing diagonal, which suggests an odd number diagonals. When we take shortcuts to make ends meet, we are always left with a gap inside the figure. Additional deficiencies arise with respect to the tessellation, where the main figure and the complement figure either overlap or leave gaps too.

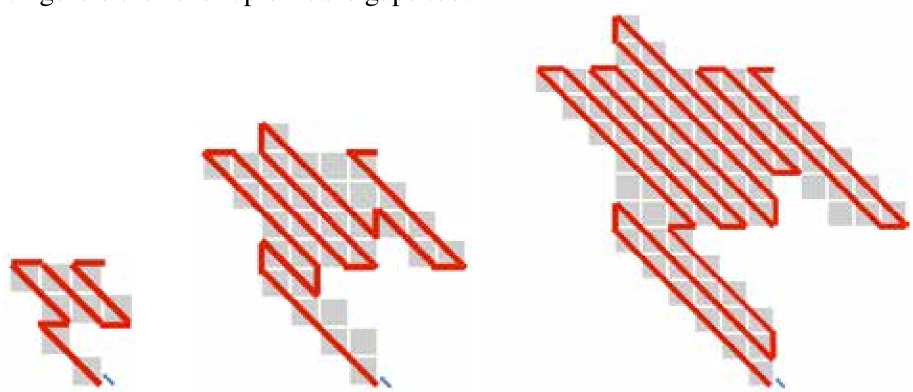


Figure 3: *Satisfactory zigzag for $N = 1$ (left) and unsatisfactory zigzags for $N = 2$ (centre), $N = 3$ (right).*

Even if we gave up on the idea that the diagonals had to be well-aligned with respect to the original grid of the classic pied de poule, we found no solutions which satisfied all the requirements of the section "requirements". We did not want to let the turtle walk back over its own trace (which would have been an ugly solution with lots of ambiguity). But at some point we took another fresh look at the concept of continuity and its opposite, discontinuity, which in turtle graphics means "pen up" and "pen down": the solution presented in the next section.

The novel fractal line pied de poule

Now consider the following idea: if we would be allowed to use pen-up and pen-down turtle graphics commands, then we could draw all the essential diagonals and connect them by special line segments and arcs. The special line segments and arcs would be outside of the classic figure, but we could draw them with pen-up and thus they would not be harmful. Or perhaps we could draw them with a very thin pen and they would be "almost" not harmful. This is shown in Figure 4.

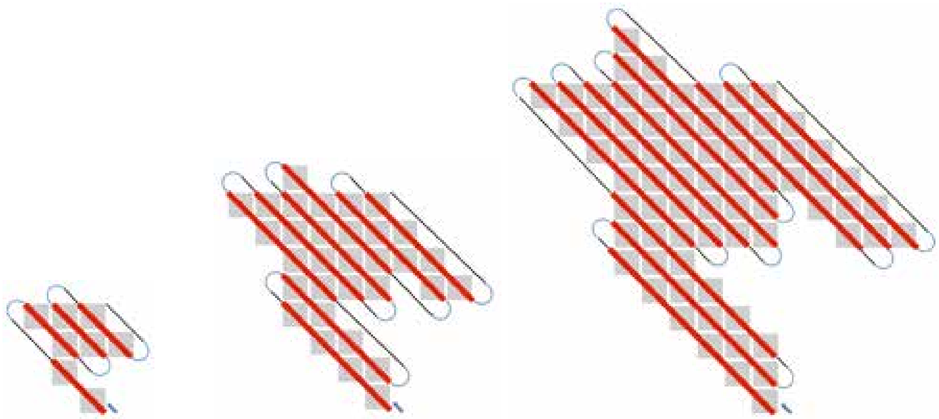


Figure 4: *Drawing the diagonals of a classic pied de poule with outer loops drawn with a thinner pen for* $N = 1$ *(left),* $N = 2$ *(center) and* $N = 3$ *(right).*

Instead of having a pen-down command, we let the drawing function work recursively, writing pied de poules all along. In order to make sure the figures tessellate correctly, we have to do two diagonal pied de poule figures in each cell. So they shrink by a factor of $\frac{1}{8}\sqrt{2}$ (for $N = 1$), by $\frac{1}{16}\sqrt{2}$ (for $N = 2$) and by $\frac{1}{24}\sqrt{2}$ (for $N = 3$). In general they shrink by a factor $\frac{1}{8N}\sqrt{2}$. The effect is demonstrated in Figure 5 for $N = 3$.

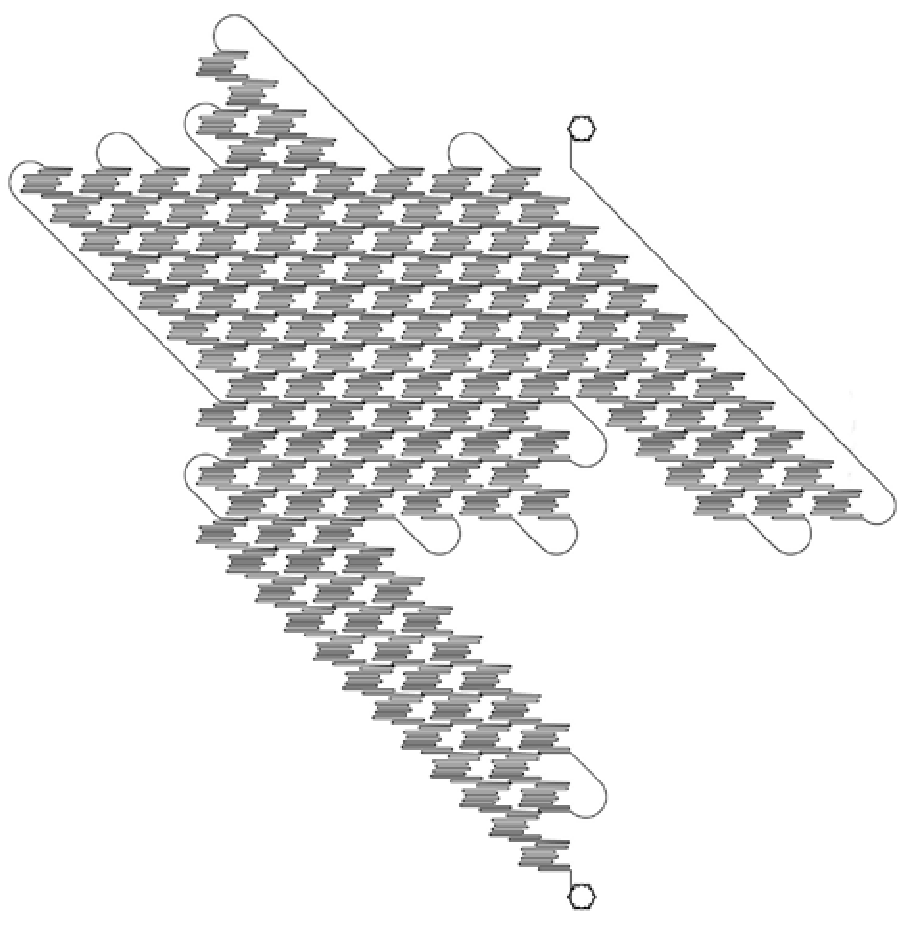


Figure 5: *Fractal line pied de poule approximation: solution for* $N = 3$ *and* $n = 2$ *based on the infinitesimally thin outer-loops concept,* © *Loe Feijs.*

The outer loops are needed to connect the zigzagged diagonals. Each outer loop consists of a line segment and an arc, which is a half circle. The choice to use half circles is made for aesthetic reasons (we tried straight lines but that appeared ugly). The half circles make the construction less edgy and are neutral with respect to directionality. We can only draw approximations, such as for example the approximation corresponding to $n = 2$ in Fig. 5. The effect is that the supposedly "thin" lines are thin indeed. We call them *infinitesimally* thin lines. In our practical approximations, they are visible (but thin indeed). The whole figure consists of a single line which does not cross itself. This solution works for all N. It is a remarkable feature of this construction that the figure rotates by 45° at level $n-1$, by 90° at $n-2$ and so on. The line touches at certain points, but we can tweak this almost invisibly and have one long non-intersecting line.

Intuitively we can say that the outer loops are a minor thing, but can we prove it in a formal sense? It turns out that adding an ε-fattening band around the classic pied de poule is enough to let it cover the fractal line pied de poule approximation, including the protruding outer loops. We find that this ε vanishes for $N \to \infty$. So we can neglect the outer loops for large N, but even for fixed N we can show that the practical drawing of the outer loops must be of a thickness going to zero when approximating for $n \to \infty$. The formulation of the formal properties and the mathematical proofs are outside the scope of this paper.

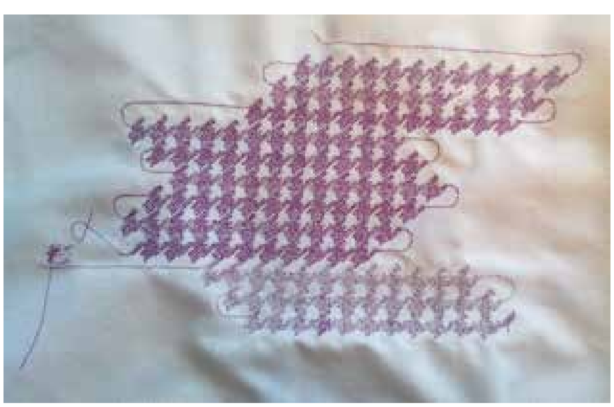
Fractal dimension

The fractal dimensions are comparable to those of the earlier Cantor-dust inspired fractals. For $N = 1$ we refer to Fig. 4 (left). Replacing a line segment of length 1 by a zigzagged pied de poule we replace it by 16 'line' segments of length $\frac{1}{8}\sqrt{2}$ each. Writing m for the number of line segments, s for the scaling factor, $m = 16$ and $\frac{1}{s} = 1/(\frac{1}{8}\sqrt{2}) = 4\sqrt{2}$ so the dimension $D(1) = (\log m)/(\log \frac{1}{s}) = (\log 16)/(\log 4\sqrt{2}) = 4/2\frac{1}{2} = 1.6$. For $N = 2$ we find $m = 64$ and $\frac{1}{s} = 1/(\frac{1}{16}\sqrt{2}) = 8\sqrt{2}$ so $D(2) = (\log 64)/(\log 8\sqrt{2}) = 6/3\frac{1}{2} = 1.7143$. For $N = 3$ we find $m = 144$ and $\frac{1}{s} = 1/(\frac{1}{24}\sqrt{2}) = 12\sqrt{2}$ so $D(3) = (\log 144)/(\log 12\sqrt{2}) = 1.7552$.

In general we find $m = (4 \times N)^2$ and $\frac{1}{s} = 1/(\frac{1}{8N}\sqrt{2}) = 4N\sqrt{2}$ so $D_N = (2\log 4N)/(\log 4N\sqrt{2})$. Compare these results with the earlier fractal pied de poule (called fPDP) [2]: $D_{fPDP}(1) = 1.5$, $D_{fPDP}(2) = 1.6667$, and $D_{fPDP}(3) = 1.7211$. Again $\lim_{N\to\infty} D(N) = 2$ but convergence is slow as for $D(N) = 1.99$ we need $N = 10^{30}$.

Application

Once we had found a satisfying fractal line pied de poule we considered the medium to apply it: embroidery and/or laser engraving (see Fig. 6).



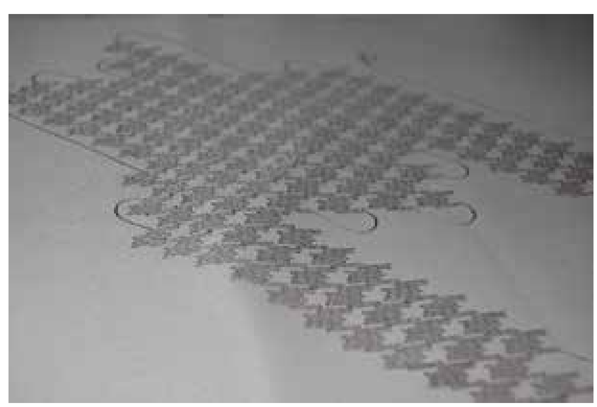


Figure 6: *Embroidered sample (left) and laser-engraved sample (right).*

Modern embroidery machines can read the stitches from a file. We programmed a layer of Processing (Java) software around the open-source turtle graphics library Oogway [4], which we named Stitchway. It supports turtle graphics like Oogway and besides generates a file in the ternary Tajima file format, which can be interpreted by Brother embroidery machines and embroidery simulators such as TrueSizer by Wilcom. Then we made both embroidered and laser-cut samples. A first sample of embroidered textile is in Fig. 6 (left) and another sample engraved in multi-layer woven textile is in Fig. 6 (right).

We have designed and constructed a mini-collection of three attractive high-tech fashionable garments based on the new fractal line pied de poule:

- a body stocking,
- a parka (coat), and
- a jacket.

The body stocking and the parka can be seen in Fig. 7 (left) whereas the body stocking and the jacket can be seen in Fig. 7 (right). A larger photograph of the parka is in Fig. 8.

Figure 7: *Fractal pied de poule collection 2015: Body stocking and Parka (left), Body stocking and jacket (right). Lasercut multilayer woven polyester fabric, model Renata van Putten, photo Brian Smeulders, © Marina Toeters.*

Conclusions

The search for this fractal was quite an adventure: the attempts illustrated in Fig. 3 are just a few from an almost endless series of failing attempts to solve the puzzle (for which we did not know beforehand whether there was a solution waiting for us). We consider the zigzags with infinitesimally thin outer loops a very elegant solution. It allows us to completely avoid the ad-hoc clumsyness of pen-up/down commands or line thicknesses. Yet in the theoretical sense of $N \to \infty$, the zigzags do not protrude out of the classic pied de poule figure. Moreover, in another theoretical sense, viz. approximating for $n \to \infty$, the outer loops are invisible anyhow. The solution has no further ambiguities: if we formally define a *zigzag* to be any alternation of straight lines and arcs (semicircles) without sharp bends, then there is a unique zigzag of minimal length which covers the diagonals of the classic pied de poule beginning and ending at the connector points of the classic chain. Inside the classic figure the pen is down, meaning there is recursion; outside the pen is up, that is: no recursion.

Practically, this solution allows us to choose any continuous medium such as stitching, embroidery, plotting (without lifting the pen), bended wire, bended pipes and so on. Using wire, conductive yarn, or conductive ink, the figure turns into a resistor, or an antenna, or even a sensor for touch[1]. The novel line fractal pied de poule yields figures with dimensions which are very close to those of our earlier fractal [2], even though the former begins from a one-dimensional figure (the line) which is extended by zigzagging whereas the latter begins as a two-dimensional figure (a classic pied de poule) which is Cantor-dustified by cutting out subsquares. But the fractals are *not* the same, for example the 45° rotation is a unique feature of the novel construction. Finally, we thank Jun Hu, Lilian Admiraal, Chet Bangaru, Jasper Sterk and Jan Rouvroye for their kind support.

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[1] In fact our long term ambition is to combine the experience we have gathered in this project with other sensor and actuator projects into a more complex and interactive system (our ambition for next year's Bridges).

Figure 8: *Fractal pied de poule collection 2015: Parka. Lasercut multilayer woven polyester fabric, model Renata van Putten, photo Brian Smeulders, © Marina Toeters.*